ISRAEL JOURNAL OF MATHEMATICS 117 (2000), 261–284

LANGLANDS–SHAHIDI METHOD AND POLES OF AUTOMORPHIC L-FUNCTIONS II

BY

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ABSTRACT

We use Langlands–Shahidi method and the observation that the local components of residual automorphic representations are unitary representations, to study the Rankin–Selberg L-functions of $\mathrm{GL}_k \times$ classical groups. Especially we prove that $L(s, \sigma \times \tau)$ is holomorphic, except possibly at $s = 0, \frac{1}{2}, 1$, where σ is a cuspidal representation of GL_k which satisfies weak Ramanujan property in the sense of Cogdell and Piatetski-Shapiro and τ is any generic cuspidal representation of SO_{2l+1}. Also we study the twisted symmetric cube L-functions, twisted by cuspidal representations of GL_2.

Introduction

In this paper we use Langlands–Shahidi method [Sh1–3] and the following observation to prove the holomorphy of several *completed* automorphic L-functions which appear in constant terms of the Eisenstein series. Because of the functional equation $L(s, \sigma, r) = \epsilon(s, \sigma, r)L(1-s, \tilde{\sigma}, r)$, it is enough to establish the holomorphy for $\mathrm{Re}\, s \geq \frac{1}{2}$.

OBSERVATION: *The local components of residual automorphic representations are unitary representations.*

It was Speh [Sp] who applied this observation to prove that certain representations of GL_n are unitary. Tadic [Ta, Appendix] adopted her method. In their

* Partially supported by NSF grant DMS9610387.

Received August 3, 1998

paper they used the global information to get local information. To the author's knowledge, this paper is the first to try to use the opposite direction, that is, to use the local information on unitary representations to get information on the location of poles of automorphic L-functions. Our results depend heavily on the classification of unitary representations.

We apply the above observation to the following situation: We follow [Sh3] and use the same notation. Let G be a quasi-split group over a number field F and $P = MN$ be a maximal parabolic subgroup. Let σ be a cuspidal representation of M and denote the residual spectrum attached to (M,σ) by $L^2_{dis}(G(F)\backslash G(\mathbb{A}))_{(M,\sigma)}$. We know that its constituents are $\pi = \otimes\pi_v$, where π_v is the "Langlands' quotient" $J(s,\sigma_v)$ of $I(s,\sigma_v)$ for some $s > 0$ (only in the case of tempered σ_v, it is the usual Langlands' quotient of $I(s,\sigma_v)$). We may and will assume that the poles of Eisenstein series may be on the real axis by normalizing σ so that the action of the maximal torus in the center of M at the archimedean places is trivial (see section 1). Here $I(s,\sigma_v) = I(s\tilde{\alpha},\sigma_v) = \mathrm{Ind}_P^G\, \sigma_v \otimes \exp(\langle s\tilde{\alpha}, H_P(\ )\rangle)$ is the induced representation (see section 1). The poles of the Eisenstein series attached to (M,σ) coincide with those of its constant term which contains automorphic L-functions and the local normalized intertwining operators. We can prove that the local normalized intertwining operators are holomorphic and non-zero for $\mathrm{Re}\, s \geq \frac{1}{2}$. So if the automorphic L-function has a pole, then the residue of the Eisenstein series belongs to the residual spectrum and thus each $J(s,\sigma_v)$ is a unitary representation. In many cases, which Langlands' quotients are unitary has been determined. By looking at when $J(s,\sigma_v)$ is unitary for an appropriate local place (for example, tempered σ_v), we can find the information on the poles of the automorphic L-functions. Here we should note the normalization in [Sh3] for $\tilde{\alpha}$; for example, for $G = \mathrm{Sp}_{2n}, P = MN, M = \mathrm{GL}_n$, $I(s\tilde{\alpha},\sigma_v) = \mathrm{Ind}_P^G(\sigma_v \otimes |\det|^s) \otimes 1$. However, for $G = \mathrm{SO}_{2n+1}$ or $\mathrm{SO}_{2n}, P = MN, M = \mathrm{GL}_n$, $I(s\tilde{\alpha},\sigma_v) = \mathrm{Ind}_P^G(\sigma_v \otimes |\det|^{s/2}) \otimes 1$. Notice $s/2$ instead of s. This is crucial in determining to which L-functions our argument can be applied to prove the holomorphy for $\frac{1}{2} \leq s < 1$.

In [Ki-Sh] we applied this method to the case when G is a split group of type G_2 over a number field and $P = MN$, $M = \mathrm{GL}_2$, where P is attached to the long simple root. Ramakrishnan [Ram] showed that any cuspidal representation of GL_2 has at least one unramified tempered local component. Actually he proves the much stronger result that more than 90% of unramified local components of a cuspidal representation of GL_2 are tempered. Muić [Mu1] has given us the classification of unitary representations of p-adic G_2. When these results were combined with several other local and global results, in [Ki-Sh] we succeeded in proving that symmetric cube L-functions $L(s,\sigma,\mathrm{Sym}^3)$ are entire for non-monomial representations. Actually we only need a weaker result that σ satisfies weak Ramanujan property in the sense of [Co-PS1] (see Definition 3.1). It is known that a cuspidal representation of $\mathrm{GL}_2, \mathrm{GL}_3$ satisfies weak Ramanujan property in the sense of [Co-PS1].

In this paper, we study the Rankin–Selberg L-functions for $\mathrm{GL}_n \times G_m$, where G_m is a split classical group, Sp_{2m}, or SO_{2m+1}. We prove

THEOREM 0.1: *Let G_m be a split classical group, Sp_{2m}, or SO_{2m+1}. Let σ (τ) be a generic cuspidal representation of GL_n (G_m). Suppose σ satisfies weak Ramanujan property in the sense of [Co-PS1]. Then*

(1) *The completed Rankin–Selberg L-function $L(s,\sigma\times\tau)$ is holomorphic for* $\mathrm{Re}\, s > 1$.
(2) *Let $G_m = \mathrm{SO}_{2m+1}$ ($m \geq 1$). Then the completed Rankin–Selberg L-function $L(s,\sigma\times\tau)$ is holomorphic except possibly at $s = 0, \frac{1}{2}, 1$.*

We use the classification of unitary representations coming from unramified principal series due to Yoshida [Yo]. The reason we get the definite result in the case of SO_{2n+1} is that there are no unitary representations if $s > \frac{1}{2}$. In order to prove that the local normalized intertwining operators are holomorphic and non-zero for $\mathrm{Re}\, s \geq \frac{1}{2}$, we need to use standard module conjecture in these cases proved in [Mu2] and also some global argument (see Lemma 3.3). The major obstacle in the case of Sp_{2n} is that we do not know whether a generic cuspidal representation of Sp_{2n} satisfies weak Ramanujan property. However, in the lower rank cases such as $\mathrm{GL}_2 \times \mathrm{SL}_2$, we get the definite result.

There are limitations in our method. For example, we cannot prove the holomorphy of the exterior square L-function $L(s,\sigma,\wedge^2)$ for $0 < s < 1$ for even n and self-contragredient σ because of normalization $s/2$ in the above. There are unitary representations for $0 < s < 1$. The same things happen with the Rankin–Selberg L function of $\mathrm{GL}_n \times \mathrm{GL}_n$ when $\sigma_1 \simeq \sigma_2$ and the symmetric square L-function $L(s,\sigma,\mathrm{Sym}^2)$.

Next we study the Rankin–Selberg L-function $L(s,\mathrm{Ad}^3(\pi)\times\pi')$, where π,π' are cuspidal representations of GL_2 and $\mathrm{Ad}^3(\pi) = \otimes_v \mathrm{Ad}^3(\pi_v)$ is the adjoint cube defined locally in [Sh6], except when π_v is an extraordinary supercuspidal representation of $\mathrm{GL}_2(F_v)$: If π_v is an extraordinary supercuspidal representation, even though we do know whether $\mathrm{Ad}^3(\pi_v)$ exists, we can still define the local L-function $L(s,\mathrm{Ad}^3(\pi_v)\times\pi_v')$ as a quotient $\dfrac{L(s,\sigma_v,r_1)}{L(s,\pi_v\times\pi_v')}$ (see (4.1)). This will be done by studying the case $D_5 - 2$ in [Sh3]. The case $D_4 - 2$ was done in [Ki-Sh2], yielding the holomorphy of the completed Rankin triple L-functions of $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$. We prove

THEOREM 0.2: *Let π, π' be two cuspidal representations of GL_2. Then $L(s, \mathrm{Ad}^3(\pi) \times \pi')L(s, \pi \times \pi')$ is holomorphic except possibly at $s = 0, \frac{1}{2}, 1$.*

In order to obtain the holomorphy of $L(s, \mathrm{Ad}^3(\pi) \times \pi')$, we need to prove that $L(s, \pi \times \pi')$ does not have zeros for $\frac{1}{2} < \mathrm{Re}\, s < 1$. This is the so-called generalized Riemann Hypothesis. However, in light of the converse theorem [Co-PS2], this gives the strongest possible evidence for the existence of the adjoint cube lift $\mathrm{Ad}^3(\pi)$ for a cuspidal representation π of GL_2.

ACKNOWLEDGEMENT: We would like to thank Prof. F. Shahidi for his constant help in explaining his results and for many discussions. Thanks are due to the referee for many comments and corrections.

1. Preliminaries

Recall several facts and notations from [Ki3]: Let $\mathbf{G}$ be a quasi-split group over a local field and $\mathbf{P} = \mathbf{MN}$ is a maximal parabolic subgroup and let α be the unique simple root in $\mathbf{N}$. As in [Sh1], let $\tilde{\alpha} = \langle \rho, \alpha \rangle^{-1} \cdot \rho$, where ρ is half the sum of roots in $\mathbf{N}$. We identify $s \in \mathbb{C}$ with $s\tilde{\alpha} \in \mathfrak{a}_{\mathbb{C}}^*$ and denote $I(s, \sigma) = I(s\tilde{\alpha}, \sigma) = \mathrm{Ind}_P^G\, \sigma \otimes \exp(\langle s\tilde{\alpha}, H_P(\ )\rangle)$.

Let $A(s\tilde{\alpha}, \sigma, w_0)$ be the standard intertwining operator from $I(s\tilde{\alpha}, \sigma)$ into $I(w_0(s\tilde{\alpha}), w_0(\sigma))$. Denote by ${}^LM$ the L-group of $\mathbf{M}$ and let ${}^L\mathfrak{n}$ be the Lie algebra of the L-group of $\mathbf{N}$. Let r be the adjoint action of ${}^LM$ on ${}^L\mathfrak{n}$ and decompose $r = \bigoplus_{i=1}^m r_i$, with ordering as in [Sh1]. For each i, $1 \le i \le m$, let $L(s, \sigma, r_i)$ be the local L-function defined in [Sh1]. It is defined to agree completely with Langlands definition of L-functions whenever there is a parametrization. In particular the L-function for arbitrary σ is just the analytic continuation of the one attached to the tempered inducing data through the product formula (cf. part 3 of Theorem 3.5 and equation 7.10 of [Sh1]). (See also Theorem 5.2 of [Sh2].)

Recall conjecture 7.1 of [Sh1]:

CONJECTURE: *Assume σ_v is tempered and generic. Then each $L(s, \sigma_v, r_i)$ is holomorphic for $\mathrm{Re}\, s > 0$.*

This conjecture is true for archimedean places [A].

PROPOSITION 1.1 ([Sh1, p. 309]): *Assume σ_v is tempered and generic.*

(1) *If $m = 1$, $L(s, \sigma_v, r)$ is holomorphic for $\mathrm{Re}\, s > 0$.*

(2) *If $m = 2$ and $L(s, \sigma_v, r_2) = \prod_j (1 - \alpha_j q_v^{-s})^{-1}$, possibly an empty product where each $\alpha_j \in \mathbb{C}$ is of absolute value one (in particular if r_2 is one-dimensional, this holds), then $L(s, \sigma_v, r_1)$ is holomorphic for $\operatorname{Re} s > 0$.*

PROPOSITION 1.2 ([Ca-Sh, p. 573]): *If G is a quasi-split classical group, then the conjecture holds.*

Now let $\mathbf{G}$ be a quasi-split group over a number field F and $\mathbf{P} = \mathbf{MN}$ be a maximal parabolic subgroup. Let σ be a cuspidal representation of $M = \mathbf{M}(\mathbb{A})$. We may and will assume that the poles of Eisenstein series may be on the real axis by assuming that σ is trivial on A part of $P(\mathbb{R})$, where $P(\mathbb{R}) = M^0AN$ is the Langlands decomposition. In the case of $M = \mathrm{GL}_n$, we can identify the A part of $P(\mathbb{R})$ with F_∞^+, where $\mathbb{A}_F^* = \mathbb{I}^1 \cdot F_\infty^+$ with $\mathbb{I}^1$ ideles of norm 1. So in this case the central character ω_σ of σ is trivial on F_∞^+.

For $f \in I(s, \sigma)$, let $E(s, f, g, P)$ be the Eisenstein series attached to (M, σ) (see [Ki3] or [Sh3, section 2] for more details). Given a parabolic subgroup $Q = M_Q N_Q$, the constant term of $E(s, f, g, P)$ along N_Q is zero if $Q \neq P$ and $Q \neq P'$. If P is not self-conjugate, then

$$E_N(s, f, g, P) = f(g), \quad E_{N'}(s, f, g, P) = M(s, \sigma, w_0) f(g).$$

If P is self-conjugate, then $E_N(s, f, g, P)$ is a sum of the above two terms. Here $M(s, \sigma, w_0)$ is the standard intertwining operator from the global induced representation $I(s, \sigma)$ to $I(w_0 s, w_0 \sigma)$. Let $M(s, \sigma, w_0) = \otimes_v A(s, \sigma_v, w_0)$. We normalize the intertwining operator $A(s, \sigma_v, w_0)$ as follows:

$$\begin{aligned} A(s, \sigma_v, w_0) &= r(s, \sigma_v, w_0) N(s, \sigma_v, w_0), \\ r(s, \sigma_v, w_0) &= \prod_{i=1}^{m} \frac{L(is, \sigma_v, r_i)}{L(1 + is, \sigma_v, r_i)\epsilon(s, \sigma_v, r_i, \psi_v)}. \end{aligned} \tag{1.1}$$

Let $N(s, \sigma, w_0) = \otimes_v N(s, \sigma_v, w_0)$, $r(s, \sigma, w_0) = \prod_v r(s, \sigma_v, w_0)$ and $\epsilon(s, \sigma, r_i) = \prod_v \epsilon(s, \sigma_v, r_i, \psi_v)$. Then we have, for $f \in I(s, \sigma)$,

$$\begin{aligned} M(s, \sigma, w_0) f &= r(s, \sigma, w_0) N(s, \sigma, w_0) f, \\ r(s, \sigma, w_0) &= \prod_{i=1}^{m} \frac{L(is, \sigma, r_i)}{L(1 + is, \sigma, r_i)\epsilon(s, \sigma, r_i)}. \end{aligned} \tag{1.2}$$

Suppose we have the following:

ASSUMPTION (A): *$N(s, \sigma_v, w_0)$ is holomorphic and non-zero for* $\mathrm{Re}\, s \geq \frac{1}{2}$ *for any v.*

Denote the image of $N(s, \sigma_v, w_0)$ by $J(s, \sigma_v)$. If σ_v is tempered, it is the usual Langlands' quotient $J(s, \sigma_v)$ (see [Ca-Sh] for precise references). But if σ_v is non-tempered, it is the Langlands' quotient coming from lower parabolic subgroups. Let $J(s, \sigma) = \bigotimes_v J(s, \sigma_v)$.

Let $\sigma = \bigotimes_v \sigma_v$ be a unitary cuspidal representation of M. Then each σ_v is a unitary representation.

OBSERVATION 1.3: *Suppose we have Assumption (A). If $r(s, \sigma, w_0)$ has a pole, then $J(s, \sigma) = \bigotimes_v J(s, \sigma_v)$ belongs to the residual spectrum $L^2_{dis}(G(F)\backslash G(\mathbb{A}))_{(M,\sigma)}$, and in particular, each $J(s, \sigma_v)$ is unitary.*

Let $\sigma = \bigotimes \sigma_v$ be a globally generic unitary cuspidal representation of M. Then for all v, σ_v is generic and unitary. Suppose σ_v is non-tempered. The following standard module conjecture is proved for various cases including GL_n. Especially it is true for archimedean places due to Vogan [V]. In [Mu2], it is proved for Sp_{2n} and SO_{2n+1} over non-archimedean places. In [Ca-Sh], it is proved for any quasi-split group when π_0 is supercuspidal.

STANDARD MODULE CONJECTURE: *Given a non-tempered, generic σ_v, there is a tempered data π_0 and a complex parameter Λ_0 which is in the corresponding positive Weyl chamber so that* $\sigma_v = I_{M_0}(\Lambda_0, \pi_0) = \mathrm{Ind}_{M_0}^M(\pi_0 \otimes q_v^{<\Lambda_0, H_{P_0}^M(\ )>})$.

Recall the following [Ki3, Lemma 2.4].

LEMMA 1.4: *If $s\tilde{\alpha} + \Lambda_0$ is in the positive Weyl chamber for* $\mathrm{Re}\, s \geq \frac{1}{2}$ *together with standard module conjecture and conjecture 7.1 of [Sh1], then Assumption (A) holds.*

Now we recall the technique in [Sh2] of showing that $\prod_{i=1}^m L(1 + is, \sigma, r_i)$ is non-vanishing on $\mathrm{Re}\, s = 0$.

Fix a non-trivial character $\psi = \bigotimes \psi_v$ of $F\backslash\mathbb{A}$. Then there is a natural generic character χ of $U(F)\backslash U(\mathbb{A})$ defined by ψ. We again use χ to denote $\chi|_{U(\mathbb{A})\cap M(\mathbb{A})}$. Then for any generic cuspidal representation σ of M, by changing the splitting in $\mathbf{M}$ we may assume that σ is χ-generic. Recall χ-Fourier coefficient of $E(s, f, g, P)$ [Sh2]:

$$E_\chi(s, f, g, P) = \int_{U(F)\backslash U(\mathbb{A})} E(s, f, ug, P)\bar{\chi}(u)\, du.$$

Since $U(F)\backslash U(\mathbb{A})$ is compact, the poles of $E_\chi(s,f,g,P)$ are among those of $E(s,f,g,P)$. For $f=\bigotimes_v f_v \in I(s,\sigma)$ and $g=e=(e_v)$, the identity element of $G(\mathbb{A})$, we have [Sh2]

$$E_\chi(s,f,e,P)=\prod_{v\notin S} W_{f_v}(s,e_v)\prod_{i=1}^{m} L_S(1+is,\sigma,r_i)^{-1},$$

where W_{f_v} is the Whittaker model of $I(s,\sigma_v)$. Then W_{f_v} is holomorphic for $\mathrm{Re}\, s>0$ and non-vanishing (see [Sh2, Proposition 3.1]). Therefore, the zeros of $\prod_{i=1}^m L_S(1+is,\sigma,r_i)$ are among the poles of the Eisenstein series $E(s,f,g,P)$. So we have

PROPOSITION 1.5 (Shahidi [Sh2]): *If the Eisenstein series $E(s,f,g,P)$ does not have a pole at s_0, i.e., there is no residual spectrum at s_0, then $\prod_{i=1}^m L_S(1+is,\sigma,r_i)$ has no zero at s_0.*

Shahidi [Sh2] showed that $\prod_{i=1}^m L_S(1+is,\sigma,r_i)$ is non-vanishing on $\mathrm{Re}\, s=0$ by using the fact that $E(s,f,g,P)$ is holomorphic on $\mathrm{Re}\, s=0$.

COROLLARY 1.6 ([Ki3, Lemma 2.3]): *If P is not self-conjugate or $w_0\sigma\not\cong\sigma$, the Eisenstein series $E(s,f,g,P)$ does not have a pole for $\mathrm{Re}\, s>0$. Hence in these cases, $\prod_{i=1}^m L_S(1+is,\sigma,r_i)$ has no zeros for $\mathrm{Re}\, s>0$.*

Remark 1.1: We should mention that in [Ki3], "$w_0\sigma=\sigma$" should be written as "$w_0\sigma\cong\sigma$", and "$w_0\sigma\neq\sigma$" as "$w_0\sigma\not\cong\sigma$".

LEMMA 1.7 ([Zh]): *Let σ_v be an irreducible tempered, generic representation of M. Then if $N(\Lambda,\sigma_v,w_0)$ is holomorphic at Λ_0 and conjecture 7.1 of [Sh1] holds, then it is non-zero at Λ_0.*

Proof: Let w_1 be a Weyl group element such that $w_1\Lambda_0$ is in the closure of the positive Weyl chamber. Consider the cocycle relation $N(w_1\Lambda_0,w_1\sigma_v,w_0w_1^{-1})=N(\Lambda_0,\sigma_v,w_0)N(w_1\Lambda_0,w_1\sigma_v,w_1^{-1})$. Here $N(w_1\Lambda_0,w_1\sigma_v,w_0w_1^{-1})$ is holomorphic and non-zero. Also $N(w_1\Lambda_0,w_1\sigma_v,w_1^{-1})$ and $N(\Lambda_0,\sigma_v,w_0)$ are both holomorphic. Therefore $N(\Lambda_0,\sigma_v,w_0)$ cannot be zero. ∎

PROPOSITION 1.8: *Let $\sigma=\bigotimes_v\sigma_v$ be a unitary, generic cuspidal representation of M. Assume standard module conjecture and conjecture 7.1 of [Sh1] so that Lemma 1.7 may be applied. Let S be a finite set of places, including all the archimedean places, such that for every $v\notin S$, σ_v is unramified. Suppose that $M(s,\sigma,w_0)$ is holomorphic for $\mathrm{Re}\, s\geq 1$, i.e., the Eisenstein series attached to σ is holomorphic for $\mathrm{Re}\, s\geq 1$. Suppose that the quotient $\prod_{i=1}^m \dfrac{L_S(is,\sigma,r_i)}{L_S(1+is,\sigma,r_i)}$ is* *holomorphic for* $\operatorname{Re} s > 1$ *and non-zero for* $\operatorname{Re} s \geq 1$, *and the local L-functions* $L(s, \sigma_v, r_i)$, $2 \leq i \leq m$, *are holomorphic for* $\operatorname{Re} s \geq 1$. *Then for each v, the normalized operator* $N(s, \sigma_v, w_0)$ *and the local L-function* $L(s, \sigma_v, r_1)$ *are holomorphic for* $\operatorname{Re} s \geq 1$.

*Proof**:* Take $f = \bigotimes_v f_v$ such that for each $v \notin S$, f_v is the unique K_v-fixed function normalized by $f_v(e_v) = 1$ and let $\tilde{f}_v$ be the K_v-fixed function in the space of $I(-s, w_0(\sigma_v))$, normalized the same way. Then (1.2) can be written as (see [Sh3, (2.7)])

$$(1.3) \qquad M(s, \sigma, w_0)f = \prod_{i=1}^{m} \frac{L_S(is, \sigma, r_i)}{L_S(1 + is, \sigma, r_i)} \bigotimes_{v \notin S} \tilde{f}_v \otimes \bigotimes_{v \in S} A(s, \sigma_v, w_0) f_v.$$

We imitate the proof of [Sh3, Theorem 5.2]. Fix $v \in S$, and normalize $A(s, \sigma_v, w_0)$. For each $u \in S$, $u \neq v$, $A(s, \sigma_u, w_0)$ is not a zero operator. Pick f_u, $u \in S$, $u \neq v$, so that $A(s, \sigma_u, w_0) f_u \neq 0$. Then the above equation is written as

$$M(s, \sigma, w_0)f = \prod_{i=1}^{m} \frac{L_S(is, \sigma, r_i)}{L_S(1 + is, \sigma, r_i)} \prod_{i=1}^{m} \frac{L(is, \sigma_v, r_i)}{L(1 + is, \sigma_v, r_i)} \bigotimes_{v \notin S} \tilde{f}_v \otimes \bigotimes_{u \in S, u \neq v} A(s, \sigma_u, w_0) f_u \otimes \frac{N(s, \sigma_v, w_0)}{\prod_{i=1}^{m} \epsilon(s, \sigma_v, r_i, \psi_v)}.$$

Now pick $N_0 \geq 1$ so large that $L(1 + s, \sigma_v, r_1)$ has no poles for $\operatorname{Re} s \geq N_0$. Then the normalized operator $N(s, \sigma_v, w_0)$ is holomorphic for $\operatorname{Re} s \geq N_0 - 1$. By Lemma 1.7, $N(s, \sigma_v, w_0)$ is non-vanishing for $\operatorname{Re} s \geq N_0 - 1$. Then by our assumptions, $L(s, \sigma_v, r_1)$ has no poles for $\operatorname{Re} s \geq N_0 - 1$. Arguing inductively, we see that $L(s, \sigma_v, r_1)$ has no poles for $\operatorname{Re} s \geq 1$. ∎

Remark 1.2: It is possible to prove that modulo standard module conjecture and conjecture 7.1 of [Sh1], the normalized intertwining operators $N(s, \sigma_v, w_0)$ and the local L-functions $L(s, \sigma_v, r_i)$, are all holomorphic for $\operatorname{Re} s \geq 1$ for non-archimedean v, without the assumption that the quotient $\prod_{i=1}^{m} \frac{L_S(is, \sigma, r_i)}{L_S(1+is, \sigma, r_i)}$ is holomorphic for $\operatorname{Re} s > 1$ and non-zero for $\operatorname{Re} s \geq 1$: We only need that $M(s, \sigma, w_0)$ has only finitely many poles for $\operatorname{Re} s > 0$ and the quotient $\prod_{i=1}^{m} \frac{L_S(is, \sigma, r_i)}{L_S(1+is, \sigma, r_i)}$ has only finitely many poles and zeros for $\operatorname{Re} s \geq 1$. We do this in a future work with Shahidi.

* Thanks are due to Prof. Shahidi who pointed out an error in an earlier version of the proof.

2. Some facts on unitary representations

In this section, we assume that F is a local field of characteristic 0. We restrict ourselves to the case of split reductive groups. Let χ be an unramified unitary character of T and $\Lambda \in \mathfrak{a}^* = X(T)_F \otimes \mathbb{R}$ and $\chi' = \Lambda \otimes \chi$. Then the induced representation $I(\Lambda, \chi) = \mathrm{Ind}_B^G \chi'$ is defined. It has the unique unramified irreducible subquotients, denoted by $\pi(\Lambda, \chi)$. Suppose Λ is in the closed positive Weyl chamber and let $\Delta_1 = \{\alpha \in \Delta |\ \Lambda \circ \alpha^\vee = 1\}$. Let $P_1 = M_1 N_1$ be the standard parabolic subgroup of G generated by the roots in Δ_1. Let π_1 be the unique irreducible spherical subrepresentation of $\mathrm{Ind}_{B\cap M_1}^{M_1} \chi$.

THEOREM 2.1 ([Li, Theorem 2.2]): *The following are equivalent:*

(1) $\chi' \circ \alpha^\vee \neq |\ |$ *for any* α,
(2) $\mathrm{Ind}_{P_1}^G \Lambda \otimes \pi_1$ *is irreducible (hence equals* $\pi(\Lambda, \chi)$*),*
(3) $\pi(\Lambda, \chi)$ *is generic.*

PROPOSITION 2.2 ([Li, Lemma 2.3]): *Let $\tilde{G}$ be G be unramified reductive groups over F, and let $\phi\colon \tilde{G} \longmapsto G$ be a central isogeny defined over F. Let $\tilde{B} = \tilde{T}\tilde{U}$ be a Borel F-subgroup of $\tilde{G}$ and assume ϕ maps $\tilde{B}, \tilde{U}, \tilde{G}(\mathcal{O})$ to B, T and $G(\mathcal{O})$, respectively. Let χ be an unramified unitary character of T. Then we can define a unitary character $\tilde{\chi} = \phi^*(\chi)$ of $\tilde{T}$ by $\tilde{\chi}(\tilde{t}) = \chi(\phi(\tilde{t}))$. Conversely, given any $\tilde{\chi}$, there will be finitely many χ such that $\tilde{\chi} = \phi^*(\chi)$. Then $\pi(\Lambda, \tilde{\chi})$ is unitary if and only if $\pi(\Lambda, \chi)$ is.*

Recall the result of Yoshida on the classification of unitary unramified representations.

THEOREM 2.3 ([Yo, Theorem B]): *Let $G = \mathrm{SO}_{2n+1}$. Let $\Lambda = a_1 e_1 + \cdots + a_n e_n$. Assume $I(\Lambda, \chi)$ is irreducible. Then it is unitarizable if and only if $|a_i| < \frac{1}{2}$ for $i = 1, \ldots, n$.*

THEOREM 2.4 ([Yo, Theorem C]): *Let $G = \mathrm{Sp}_{2n}$. Let $\Lambda = a_1 e_1 + \cdots + a_n e_n$. Assume $I(\Lambda, \chi)$ is irreducible. Then if it is unitarizable, we have $|a_i| < 1$ for $i = 1, \ldots, n$.*

THEOREM 2.5 ([Yo, Theorem 11.4]): *Let $G = \mathrm{SO}_{2n}$. Let $\Lambda = a_1 e_1 + \cdots + a_n e_n$. Assume $I(\Lambda, \chi)$ is irreducible. Then if it is unitarizable, we have $a_1 - |a_n| < 1$.*

Yoshida's result is not satisfactory for $G = \mathrm{SO}_{2n}$. In order to obtain a better result, we need Shahidi's result.

THEOREM 2.6 ([Sh3, Lemma 5.8]): *Let $MN \subset G$ be a maximal parabolic subgroup of a quasi-split group over a number field. Let σ_v be a unramified local* *component of a cuspidal representation σ of M. Then for each r_i, $L(s,\sigma_v,r_i)$ is holomorphic for* $\operatorname{Re} s \geq 1$.

COROLLARY 2.7: *Let σ_v be an unramified local component of a generic cuspidal representation σ of SO_{2n}. Write it as $\sigma_v = \pi(\Lambda,\chi)$ for $\Lambda = a_1e_1 + \cdots + a_ne_n$. Then $|a_i| < 1$ for $i = 1,\dots,n$.*

Proof: This is Corollary 5.4 of [Sh3], direct consequence of Theorem 2.6. ∎

COROLLARY 2.8: *Let σ (τ) be a generic cuspidal representation of GL_k (SO_{2l+1}). The L-function $L(s,\sigma\times\tau)$ is absolutely convergent for* $\operatorname{Re} s > \frac{3}{2}$.

We recall a proposition from Muić [Mu1, Lemma 5.1].

PROPOSITION 2.9: *Let G be any reductive F-group and $P = MN$ its F-parabolic subgroup. Denote by $\mathrm{Unr}(M)$ the group of unramified characters. For any irreducible representation π of M and $\Lambda \in \mathrm{Unr}(M)$, denote $I(\Lambda,\pi) = \mathrm{Ind}_P^G \Lambda\otimes\pi$.*

(1) *The set of Λ such that $I(\Lambda,\pi)$ has unitary subquotients, is compact.*
(2) *Let π be a hermitian representation and $I(0,\pi)$ be an irreducible unitary representation. Then π is unitary.*

We record here a useful lemma.

LEMMA 2.10: *Let F be a local field of characteristic zero, archimedean or non-archimedean. Let σ_1,σ_2 be two discrete series representations of $\mathrm{GL}_k, \mathrm{GL}_l$, respectively. Then the normalized intertwining operator $N(s,\sigma_1\otimes\sigma_2,w_0)$ is holomorphic and non-zero for* $\operatorname{Re} s > -1$.

Proof: This is a special case of [M-W, proposition I.10]. Or it follows from [Ca-Sh] by noting that

$$N(s,\sigma_1\otimes\sigma_2,w_0) = \frac{L(s+1,\sigma_1\times\sigma_2)}{L(s,\sigma_1\times\sigma_2)}A(s,\sigma_1\otimes\sigma_2,w_0).$$

By [Ca-Sh, Theorem 6.2] for archimedean places and the well-known result of Zelevinsky [Ze] for non-archimedean places, GL_n satisfies generalized injectivity and thus, by [Ca-Sh, Theorem 5.1], $\dfrac{A(s,\sigma_1\otimes\sigma_2,w_0)}{L(s,\sigma_1\times\sigma_2)}$ is entire. By Proposition 1.1, $L(s,\sigma_1\times\sigma_2)$ is holomorphic for $\operatorname{Re} s > 0$. ∎

3. Rankin–Selberg L-functions for $\mathrm{GL}_k \times G_l$

Let $G_n = \mathrm{Sp}_{2n}$, SO_{2n+1} or SO_{2n}. Let $MN \subset G_{k+l}$ be a maximal parabolic subgroup with $M = \mathrm{GL}_k \times G_l$. From the cases $(B_n), (C_n)$, and $(D_n - 1)$ in [Sh3], we have the following decomposition of the adjoint representation of ${}^LM$ on ${}^L\mathfrak{n}$:

$$r = r_1 \oplus r_2$$

where

$$\begin{aligned}
&\text{if } G = \mathrm{Sp}_{2n}\ (r_2 = 0 \text{ if } k=1), \quad r_1 = \rho_k \otimes \rho^{\mathrm{SO}}_{2l+1}, \quad r_2 = \wedge^2 \rho_k,\\
&\text{if } G = \mathrm{SO}_{2n+1}, \quad \begin{cases} r_1 = \rho_k \otimes \rho^{\mathrm{Sp}}_{2l}, \quad r_2 = \mathrm{Sym}^2 \rho_k, & \text{if } l \geq 1\\ r = r_1 = \mathrm{Sym}^2 \rho_k, & \text{if } l = 0,\end{cases}\\
&\text{if } G = \mathrm{SO}_{2n}, \quad \begin{cases} r_1 = \rho_k \otimes \rho^{\mathrm{SO}}_{2l}, \quad r_2 = \wedge^2 \rho_k, & \text{if } l \geq 4\\ r = r_1 = \wedge^2 \rho_k, & \text{if } l = 0.\end{cases}
\end{aligned}$$

Here ρ_k is the standard representation of $\mathrm{GL}_k(\mathbb{C})$, ρ^{Sp}_{2l} is the standard representation of $\mathrm{Sp}_{2l}(\mathbb{C})$, $\rho^{\mathrm{SO}}_{2l+1}$ is the standard representation of $\mathrm{SO}_{2l+1}(\mathbb{C})$, and ρ^{SO}_{2l} is the standard representation of $\mathrm{SO}_{2l}(\mathbb{C})$.

Recall the following definition from [Co-PS1].

Definition 3.1: Let π be an automorphic representation. We say that π satisfies weak Ramanujan property if there exists an infinite sequence of places v_m such that

(1) the local components π_{v_m} are unramified with Satake eigenvalues $\{\lambda_{v_m,i}\}$ and
(2) for every $\epsilon > 0$ we have $\max_i\{|\lambda_{v_m,i}|, |\lambda^{-1}_{v_m,i}|\} = O(q^{\epsilon}_{v_m})$.

It is known that a cuspidal representation of $\mathrm{GL}_2, \mathrm{GL}_3$ satisfies the weak Ramanujan property.

THEOREM 3.1: *Let $MN \subset \mathrm{SO}_{2n+1}$ or SO_{2n}, $M = \mathrm{GL}_n$. Let σ be a unitary cuspidal representation of GL_n.*

(1) *If σ is not self-contragredient, then the completed L-function $L(s, \sigma, \mathrm{Sym}^2)$ and $L(s, \sigma, \wedge^2)$ have no zeros for $\mathrm{Re}\, s \geq 1$ and are entire.*
(2) *If σ is self-contragredient, then $L(s, \sigma \otimes \omega, \mathrm{Sym}^2)$ and $L(s, \sigma \otimes \omega, \wedge^2)$ have no zeros for $\mathrm{Re}\, s \geq 1$ and are entire, where ω is any non-quadratic grössencharacter of F. (If n is odd, then $L(s, \sigma, \wedge^2)$ is entire and has no zeros for $\mathrm{Re}\, s \geq 1$ [Ki3].)*
(3) *For all v, non-archimedean or archimedean, the local L-functions $L(s, \sigma_v, \mathrm{Sym}^2)$ and $L(s, \sigma_v, \wedge^2)$ are holomorphic for $\mathrm{Re}\, s \geq 1$.*

(4) *Suppose σ is self-contragredient and satisfies weak Ramanujan property. Then the completed L-functions $L(s,\sigma,\mathrm{Sym}^2)$ and $L(s,\sigma,\wedge^2)$ are holomorphic for $s>1$.*

Proof: Note that $w_0(\sigma)=\tilde{\sigma}$. Let $\sigma'=\sigma$ if σ is not self-contragredient and $\sigma'=\sigma\otimes\omega$ if σ is self-contragredient, where ω is a non-quadratic grössencharacter. Then by Corollary 1.6, the partial L-functions $L_S(s,\sigma',\mathrm{Sym}^2)$ and $L_S(s,\sigma',\wedge^2)$ have no zeros for $\mathrm{Re}\, s\geq 1$.

Consider (1.3). Since σ' is not self-contragredient, $M(s,\sigma',w_0)$ is holomorphic for $\mathrm{Re}\, s>0$. Since the unnormalized operators $A(s,\sigma'_v,w_0)$ are non-zero operators, we conclude that the partial L-function $L_S(s,\sigma',\mathrm{Sym}^2)$ and $L_S(s,\sigma',\wedge^2)$ are holomorphic for $\mathrm{Re}\, s>0$. Proposition 1.8 implies that each local L-function $L(s,\sigma'_v,\mathrm{Sym}^2)$ and $L(s,\sigma'_v,\wedge^2)$ are holomorphic for $\mathrm{Re}\, s\geq 1$. If σ is self-contragredient, given a place v, take a non-quadratic grössencharacter ω so that $\omega_v=1$. Then (3) follows.

In order to prove the statements about the completed L-functions in (1) and (2), note that Assumption (A) holds in these cases, i.e., $N(s,\sigma_v,w_0)$ is holomorphic and non-zero for $\mathrm{Re}\, s\geq \frac{1}{2}$ for all v ([Ki3, Proposition 3.4]). Hence in (1.2), $\dfrac{L(s,\sigma',\mathrm{Sym}^2)}{L(1+s,\sigma',\mathrm{Sym}^2)}$ and $\dfrac{L(s,\sigma',\wedge^2)}{L(1+s,\sigma',\wedge^2)}$ are holomorphic for $s\geq\frac{1}{2}$. Therefore $L(s,\sigma',\mathrm{Sym}^2)$ and $L(s,\sigma',\wedge^2)$ are holomorphic for $s\geq\frac{1}{2}$. We apply the functional equations.

In order to prove (4), let σ_v be an unramified local component. Then

$$\sigma_v=\mathrm{Ind}_B^{\mathrm{GL}_n}(\mu_1|\ |^{\alpha_1}\otimes\cdots\otimes\mu_r|||^{\alpha_r}\otimes\nu_1\otimes\cdots\otimes\nu_p\otimes\mu_r|\ |^{-\alpha_r}\otimes\cdots\otimes\mu_1|\ |^{-\alpha_1}),$$
$$I(s,\sigma_v)=\mathrm{Ind}_B^G\,(\mu_1|\ |^{\frac{s}{2}+\alpha_1}\otimes\cdots\otimes\mu_r|\ |^{\frac{s}{2}+\alpha_r}\otimes\nu_1^{\frac{s}{2}}\otimes\cdots\otimes\nu_p^{\frac{s}{2}}\otimes\mu_r|\ |^{\frac{s}{2}-\alpha_r}$$
$$\otimes\cdots\otimes\mu_1|\ |^{\frac{s}{2}-\alpha_1}).$$

From the weak Ramanujan property of cuspidal representations of GL_n, given $\mathrm{Re}\, s_0>1$, we can find a local component σ_v such that $\mathrm{Re}\, s_0-2\alpha_1>1$. Then by Theorem 2.1, $I(s,\sigma_v)$ is irreducible for all $\mathrm{Re}\, s\geq\mathrm{Re}\, s_0$, hence it cannot be unitary. So by Observation 1.3, there is no residual spectrum for $\mathrm{Re}\, s>1$. Consider (1.3). Since the unnormalized operators $A(s,\sigma_v,w_0)$ are non-zero operators and $L_S(s,\sigma,\mathrm{Sym}^2)$ and $L_S(s,\sigma,\wedge^2)$ are holomorphic for $\mathrm{Re}\, s>2$ by [Sh3, Theorem 5.1], $L_S(s,\sigma,\mathrm{Sym}^2)$ and $L_S(s,\sigma,\wedge^2)$ are holomorphic for $\mathrm{Re}\, s>1$. We apply (3). ∎

Remark 3.1: The holomorphy and non-vanishing of $L(s,\sigma,\mathrm{Sym}^2)$ and $L(s,\sigma,\wedge^2)$ for $\mathrm{Re}\, s>1$ follow immediately from the absolute convergence of the two L-functions for $\operatorname{Re} s > 1$. D. Bump and D. Ginzburg proved that the partial L-function $L_S(s, \sigma, \mathrm{Sym}^2)$ is holomorphic except possibly at $s = 1$. (*Symmetric square L-functions on* $\mathrm{GL}(r)$, Annals of Mathematics **136** (1992), 137–205.)

THEOREM 3.2: *Let $MN \subset G_{k+l}$, $M = \mathrm{GL}_k \times G_l$ with $l \neq 0$ and $G_l = \mathrm{Sp}_{2l}$, or SO_{2l+1}. Let σ (τ) be a unitary, generic cuspidal representation of GL_k (G_l).*

(1) *If σ is not self-contragredient, then the completed L-function $L(s, \sigma \times \tau)$ has no zeros for $\operatorname{Re} s \geq 1$ and is holomorphic for $s > 1$.*
(2) *If σ is self-contragredient, then $L(s, (\sigma \otimes \omega) \times \tau)$ has no zeros for $\operatorname{Re} s \geq 1$ and is holomorphic for $s > 1$, where ω is any non-quadratic grössencharacter of F.*
(3) *For all v, non-archimedean or archimedean, the local L-function $L(s, \sigma_v \times \tau_v)$ is holomorphic for $\operatorname{Re}(s) \geq 1$.*
(4) *Suppose σ is self-contragredient and satisfies weak Ramanujan property. Then the completed L-functions $L(s, \sigma \times \tau)$ are holomorphic for $s > 1$.*

Proof: Note that $w_0(\sigma \otimes \tau) = \tilde{\sigma} \otimes \tau$. Let $\sigma' = \sigma$ if σ is not self-contragredient and $\sigma' = \sigma \otimes \omega$ if σ is self-contragredient, where ω is a non-quadratic grössencharacter. Then by Corollary 1.6, the partial L-functions $L_S(s, \sigma' \times \tau)$ has no zeros for $\operatorname{Re} s \geq 1$, since the partial L-functions $L_S(s, \sigma', \mathrm{Sym}^2)$ and $L_S(s, \sigma', \wedge^2)$ are holomorphic for $\operatorname{Re} s > 0$ (Theorem 3.1).

Consider (1.3). Since σ' is not self-contragredient, $M(s, \sigma' \times \tau, w_0)$ is holomorphic for $\operatorname{Re} s > 0$. Since the unnormalized operators $A(s, \sigma'_v, w_0)$ are non-zero operators and $L_S(s, \sigma', \mathrm{Sym}^2)$ and $L_S(s, \sigma', \wedge^2)$ have no zeros for $\operatorname{Re} s > 1$, we conclude that the partial L-function $L_S(s, \sigma' \times \tau)$ is holomorphic for $\operatorname{Re} s > \frac{1}{2}$.

Now we apply Proposition 1.8 to $\sigma' \otimes \tau$. Standard module conjecture and conjecture 7.1 of [Sh1] are proved in [Mu2], [V] and [Ca-Sh]. Hence Proposition 1.8 implies that each local L-function $L(s, \sigma'_v \times \tau_v)$ is holomorphic for $\operatorname{Re} s \geq 1$. If σ is self-contragredient, given a place v, take a non-quadratic grössencharacter ω so that $\omega_v = 1$. Then (3) follows and so do the statements about the completed L-functions in (1) and (2).

In order to prove (4), let $\sigma_v \otimes \tau_v$ be an unramified local component. Then τ_v is a component of $\mathrm{Ind}_B^{G_l}(\eta_1| \;|^{\beta_1} \otimes \cdots \otimes \eta_l| \;|^{\beta_l})$. Then $I(s, \sigma_v \otimes \tau_v)$ is a component of $\mathrm{Ind}_B^G\,(\mu_1| \;|^{s+\alpha_1} \otimes \cdots \otimes \mu_r| \;|^{s+\alpha_r} \otimes \nu_1| \;|^{s} \otimes \cdots \nu_p| \;|^{s} \otimes \mu_r| \;|^{s-\alpha_r} \otimes \cdots \mu_1| \;|^{s-\alpha_1} \otimes \eta_1| \;|^{\beta_1} \otimes \cdots \otimes \eta_l| \;|^{\beta_l})$. Applying the same technique as in Theorem 3.1, we see that weak Ramanujan property and Theorems 2.3 and 2.4 imply that the partial L-functions $L_S(s, \sigma \times \tau)$ is holomorphic for $\operatorname{Re} s > 1$. We apply (3). ∎

In the following let $G_n = \mathrm{Sp}_{2n}$ or SO_{2n+1}. Let σ (τ) be a unitary cuspidal representation of GL_k ($G_l = \mathrm{Sp}_{2l}$ or SO_{2l+1}, $l \neq 0$). By standard module conjecture proved in [Mu2] for non-archimedean places and [V] for archimedean places, non-tempered unitary representations σ_v, τ_v of GL_k and G_l, resp. can be written as follows:

$$\sigma_v = \mathrm{Ind}(|\det|^{\alpha_1}\sigma_1 \otimes \cdots \otimes |\det|^{\alpha_p}\sigma_p \otimes \sigma_{p+1} \otimes |\det|^{-\alpha_p}\sigma_p \otimes \cdots \otimes |\det|^{-\alpha_1}\sigma_1)$$

and

$$\tau_v = \mathrm{Ind}(|\det|^{\beta_1}\tau_1 \otimes \cdots \otimes |\det|^{\beta_q}\tau_q \otimes \tau_0),$$

where $0 < \alpha_p < \cdots < \alpha_1 < \frac{1}{2}$, $0 < \beta_q < \cdots < \beta_1$, and σ_i's, τ_j, $i = 1, ..., q$, are tempered representations of GL, τ_0 is a tempered representation of G_r. Then

$$(3.1) \quad I(s, \sigma_v \otimes \tau_v) = \mathrm{Ind}\,(|\det|^{s+\alpha_1}\sigma_1 \otimes \cdots \otimes |\det|^{s+\alpha_p}\sigma_p \otimes |\det|^{s}\sigma_{p+1} \otimes |\det|^{s-\alpha_p}\sigma_p \otimes \cdots \otimes |\det|^{s-\alpha_1}\sigma_1 \otimes |\det|^{\beta_1}\tau_1 \otimes \cdots \otimes |\det|^{\beta_q}\tau_q \otimes \tau_0).$$

LEMMA 3.3: *Let σ (τ) be a cuspidal representation of GL_k ($G_l = \mathrm{Sp}_{2l}$ or SO_{2l+1}) and σ_v and τ_v be as above. Then $\beta_1 < 1$.*

Proof: We write $\tau_v = \mathrm{Ind}(|\det|^{a_1}\rho_1 \otimes \cdots \otimes |\det|^{a_r}\rho_r \otimes \rho_0)$, where $\rho_1, \ldots, \rho_r$ are discrete series representations of GL_{n_i}, $i = 1, \ldots, r$ and ρ_0 is a tempered representation of G_{n_0} and $0 < a_r \leq \cdots \leq a_1$.

If v is non-archimedean, by [Ro, Proposition 5.15], there exists a cuspidal representation π of GL_{n_1} such that $\pi_v = \rho_1$. By Theorem 3.2, $L(s, \pi_v \times \tau_v)$ is holomorphic for $\mathrm{Re}\, s \geq 1$. However, $L(s, \pi_v \times \tau_v)$ contains a factor $L(s - a_1, \rho_1 \times \rho_1)$ which has a pole at $s - a_1 = 0$. It follows $a_1 < 1$.

Next let v be an archimedean place. A discrete series for GL_n exists only for $\mathrm{GL}_1(\mathbb{R}), \mathrm{GL}_1(\mathbb{C})$ and $\mathrm{GL}_2(\mathbb{R})$. Given a unitary character ρ for $\mathrm{GL}_1(F_v)$, $F_v = \mathbb{R}, \mathbb{C}$, there exists a grössencharacter μ of F such that $\mu_v = \rho$. By Jacquet–Langlands correspondence [Ja-La], given a discrete series ρ of $\mathrm{GL}_2(F_v)$, $F_v = \mathbb{R}$, there exists a cuspidal representation τ of GL_2 such that $\tau_v = \rho$. Proceeding as in non-archimedean case, we obtain the result. ∎

PROPOSITION 3.4: *Assumption (A) holds in these cases, i.e., for all v, $N(s, \sigma_v \otimes \tau_v, w_0)$ is holomorphic and non-zero for $\mathrm{Re}\, s \geq \frac{1}{2}$.*

Proof: By Lemma 3.3, $\beta_1 < 1$ and thus in (3.1), $\mathrm{Re}(s - \alpha_1 - \beta_1) > -1$ for $\mathrm{Re}\, s \geq \frac{1}{2}$. The rank-one operators are either operators for the case $\mathrm{GL}_k \times \mathrm{GL}_l \subset \mathrm{GL}_{k+l}$ or $\mathrm{GL}_l \subset G_l$. The operator for the case $\mathrm{GL}_k \times \mathrm{GL}_l \subset \mathrm{GL}_{k+l}$ is holomorphic for $\mathrm{Re}\, s \geq \frac{1}{2}$ by Lemma 2.10. For the operator for the case $\mathrm{GL}_l \subset G_l$, we proceed as in [Ki3, Proposition 3.3, 3.4]. Hence $N(s, \sigma_v, w_0)$ is holomorphic for $\mathrm{Re}\, s \geq \frac{1}{2}$.

Applying Lemma 1.7 to (3.1), we see that $N(s,\sigma_v,w_0)$ is non-zero for $\mathrm{Re}\, s \geq \frac{1}{2}$. ∎

We look at a special case: $G = \mathrm{SO}_{2n+1}$, $MN \subset \mathrm{SO}_{2n+1}$, $M = \mathrm{GL}_k \times \mathrm{SO}_{2l+1}$, $k+l=n$. In this case, we can get the definite result due to Theorem 2.3.

THEOREM 3.5: *Let σ (τ) be a generic cuspidal representation of GL_m (SO_{2n+1}). Suppose that σ satisfies weak Ramanujan property. Then the Rankin–Selberg L-function $L(s,\sigma\times\tau)$ is holomorphic except possibly at $s=0,\frac{1}{2},1$.*

Proof: If $s>1$, $L(s,\sigma\times\tau)$ is holomorphic by Theorem 3.2. Given $\frac{1}{2}<s<1$, from weak Ramanujan property of cuspidal representations of GL_n, we can find an unramified component $\sigma_v\otimes\tau_v$ such that $s-\alpha_1>\frac{1}{2}$, $s+\alpha_1<1$. Since τ_v is unitary and unramified, by Theorem 2.3, $\beta_1<\frac{1}{2}$. Then from Theorem 2.1, $I(s,\sigma_v\otimes\tau_v)$ is irreducible, and hence is not unitary by Theorem 2.3. Therefore, $L(s,\sigma\times\tau)L(2s,\sigma,\mathrm{Sym}^2)$ is holomorphic except possibly at $s=\frac{1}{2},1$ if $s>0$; no poles if σ is not self-contragredient. Here $L(s,\sigma,\mathrm{Sym}^2)$ has no zeros for $\mathrm{Re}\, s\geq\frac{1}{2}$; on the line, $\mathrm{Re}\, s=\frac{1}{2}$, it follows from Prop. 1.5. For $\mathrm{Re}\, s>\frac{1}{2}$, consider $L(2s,\sigma\times\sigma)=L(2s,\sigma,\mathrm{Sym}^2)L(2s,\sigma,\wedge^2)$ and apply Theorem 3.1 (4) and the fact that $L(2s,\sigma\times\sigma)$ has no zeros for $\mathrm{Re}\, s>1$. ∎

For $G=\mathrm{Sp}_{2n}$ or SO_{2n}, we do not have a precise result. For SO_{2n}, we do not even have standard module conjecture for ramified places.

LEMMA 3.6: *Let $\mu_1,\ldots,\mu_n$ be unitary local unramified characters of $F^\times$ and let $\pi_1=\mathrm{Ind}_B^{\mathrm{GL}_k}(\mu_1\otimes\cdots\otimes\mu_k)$. We assume that $k\geq 2$ and π_1 has the trivial central character, i.e., $\mu_1\cdots\mu_k=1$. Let π_2 be the unique generic component of $\mathrm{Ind}_B^{G_l}(\mu_{k+1}\otimes\cdots\otimes\mu_n)$, where $G_l=\mathrm{Sp}_{2l},\mathrm{SO}_{2l}$. Then if $s>\frac{1}{2}$, $s\neq 1$, $I=\mathrm{Ind}_{\mathrm{GL}_k\times G_l}^{G_n}(|\det|^s\pi_1\otimes\pi_2)$ is irreducible and is not unitary.*

Proof: By Theorem 2.1, if $s>\frac{1}{2}$, $s\neq 1$, I is irreducible. Therefore I cannot be unitary for $s>1$. Suppose $\frac{1}{2}<s<1$ and I is unitary. Note that I is hermitian if and only if $\pi_1\simeq\tilde{\pi}_1$ and π_2 is fixed by c_n, the sign change when k is odd and $G_l=\mathrm{SO}_{2l}$. Since $\pi_1\simeq\tilde{\pi}_1$ and $\mu_1\cdots\mu_k=1$, $\mu_i=\mu_j^{-1}$ for $i\neq j$. We assume that $\mu_1=\mu_2^{-1}$. Then

$$\begin{aligned} I &\simeq \mathrm{Ind}_{F^\times\times\cdots\times F^\times\times G_l}^{G_n}(|\ |^s\mu_1\otimes\cdots\otimes|\ |^s\mu_k\otimes\pi_2)\\ &\simeq \mathrm{Ind}_{F^\times\times\cdots\times F^\times\times G_l}^{G_n}(|\ |^s\mu_1\otimes|\ |^{-s}\mu_1\otimes|\ |^s\mu_3\cdots\otimes|\ |^s\mu_k\otimes c_n(\pi_2))\\ &\simeq \mathrm{Ind}_{\mathrm{GL}_2\times G_{n-2}}^{G_n}(\pi(|\ |^s\mu_1,|\ |^{-s}\mu_1)\otimes\\ &\qquad \mathrm{Ind}_{F^\times\times\cdots\times F^\times\times G_l}^{G_{n-2}}(|\ |^s\mu_3\otimes\cdots\otimes|\ |^s\mu_k\otimes c_n(\pi_2))). \end{aligned}$$

Therefore, by Proposition 2.9, $\pi(|\ |^s\mu_1, |\ |^{-s}\mu_1)$ is unitary. However, it is not unitary for $s > \frac{1}{2}$. ∎

Recall the definition of density from [Ram].

Definition 3.2: Let S be a set of primes in a number field F. Then the upper (resp. lower) Dirichlet density of S is given by

$$\bar{\delta}(S) = \overline{\lim_{s\to 1^+}} -\frac{\sum_{v\in S} q_v^{-s}}{\log(s-1)} \quad \left(\text{resp. } \underline{\delta}(S) = \underline{\lim_{s\to 1^+}} -\frac{\sum_{v\in S} q_v^{-s}}{\log(s-1)}\right).$$

One says that S has a density, denoted by $\delta(S)$, when the upper and lower densities are equal. For example, when S is the set of all but a finite number of primes, it has a density with $\delta(S) = 1$. Note that if $X = S \cup T$ is a disjoint partition of sets of primes, then $\underline{\delta}(S) \geq \underline{\delta}(X) - \bar{\delta}(T)$.

Let $\tau = \bigotimes_v \tau_v$ be a generic cuspidal representation of Sp_{2l}. We need a condition that is a little stronger than weak Ramanujan property, i.e., given $\epsilon > 0$, the set of primes such that $\max_i\{|\lambda_{vi}|, |\lambda_{vi}^{-1}|\} \geq q_v^{\epsilon})$ has density zero, where $\{\lambda_{vi}, i = 1, \dots, n\}$ is Satake eigenvalues of τ_v for unramified places.

THEOREM 3.7: *Let σ (τ) be a generic cuspidal representation of GL_k ($G_l = \mathrm{Sp}_{2l}$). Suppose σ and τ satisfy the above condition. Then the completed L-function $L(s, \sigma \times \tau)$ is holomorphic except possibly at $s = 0, \frac{1}{2}, 1$.*

Proof: First of all, we prove that if $L(s, \sigma \times \tau)$ has a pole for $s \geq \frac{1}{2}$, then $\omega_\sigma^2 = 1$, where ω_σ is the central character of σ: By Proposition 3.4, $N(s, \sigma_v, w_0)$ is holomorphic and non-zero for $\mathrm{Re}\, s \geq \frac{1}{2}$ for all v. So by Observation 1.3, if $L(s, \sigma \times \tau)$ has a pole for $s \geq \frac{1}{2}$, then the residual automorphic representation $J(s, \sigma \otimes \tau) = \bigotimes_v J(s, \sigma_v \otimes \tau_v)$ is unitary. By assumption, in (3.1), $I(s, \sigma_v \otimes \tau_v)$ is an induced representation in the closure of the positive Weyl chamber except for places of density zero. Then in order that $J(s, \sigma_v \otimes \tau_v)$ be unitary, $J(s, \sigma_v \otimes \tau_v)$ should be hermitian and therefore, $\omega_{\sigma_v}^2 = 1$, especially for places of lower density$>\frac{1}{2}$. So by Hecke's theorem [cf. Ki-Sh], $\omega_\sigma^2 = 1$. Or this follows immediately from the fact that if $\omega_\sigma^2 \neq 1$, then $w_0(\sigma \otimes \tau) \not\simeq \sigma \otimes \tau$, and hence the Eisenstein series attached to $\sigma \otimes \tau$ is holomorphic for $s > 0$.

Then $\omega_{\sigma_v} = 1$ for places of density $\frac{1}{2}$. By Lemma 3.6, if $\sigma_v \otimes \tau_v$ is tempered, $J(s, \sigma_v \otimes \tau_v)$ cannot be unitary for $s > \frac{1}{2}$, $s \neq 1$. However, by Proposition 2.9, it cannot be unitary for non-tempered $\sigma_v \otimes \tau_v$ with Satake parameter $|\lambda_{v,i}| \leq q_v^\epsilon$ for some $\epsilon > 0$. By assumption, it is the case except for places of density zero.

Therefore $L(s, \sigma \times \tau)$ cannot have a pole for $s \geq \frac{1}{2}$, $s \neq 1$. We apply the functional equation. ∎

COROLLARY 3.8: *The completed L-function $L(s, \sigma \times \tau)$ of $\mathrm{GL}_2 \times \mathrm{SL}_2$ is holomorphic except possibly at $s = 0, \frac{1}{2}, 1$.*

Proof: Let $\pi = \bigotimes_v \pi_v$ be a cuspidal representation of GL_2 and $\{\alpha_v, \beta_v\}$ be a Satake parameter of π_v for unramified places. Then by [Ram, Lemma 3.1], given $\epsilon > 0$, the set of primes such that $|\alpha_v + \beta_v| \geq q_v^\epsilon$ has density zero. Note that if π_v is tempered, $|\alpha_v + \beta_v| \leq 2$. If π_v is non-tempered, then $|\alpha_v + \beta_v| = q_v^r + q_v^{-r}$ for some $0 < r < \frac{1}{2}$. Hence the condition in Theorem 3.7 is satisfied. ∎

4. Rankin–Selberg L-function $L(s, \mathrm{Ad}^3(\pi) \times \pi')$

Let $\pi = \bigotimes_v \pi_v$ be a cuspidal representation of GL_2. Then we can define the local adjoint cube lift of π_v [Sh6, section 3], except when π_v is an extraordinary supercuspidal representation. More specifically, $\Sigma_v = \mathrm{Ad}^3(\pi_v)$ is an admissible representation of $\mathrm{GL}_4(F_v)$ such that

(1) if v is archimedean, then Σ_v is an irreducible admissible representation of $\mathrm{GL}_4(F_v)$ attached to $\mathrm{Ad}^3 \circ\varphi$: $W_F \longrightarrow \mathrm{GL}_4(\mathbb{C})$, where φ: $W_F \longrightarrow \mathrm{GL}_2(\mathbb{C})$ is the homomorphism attached to π_v;

(2) if v is non-archimedean and $\pi_v = \pi(\mu, \nu)$, where μ, ν are unitary characters of $F_v^\times$, then $\Sigma_v = \mathrm{Ind}_B^{\mathrm{GL}_4(F_v)}(\mu^2\nu^{-1} \otimes \mu \otimes \nu \otimes \mu^{-1}\nu^2)$;

(3) if v is non-archimedean and $\pi_v = \pi(\mu|\ |^r, \mu|\ |^{-r})$, where μ is a unitary characters of $F_v^\times$ and $0 < r < \frac{1}{2}$, then Σ_v is a unique quotient of $\mathrm{Ind}_B^{\mathrm{GL}_4(F_v)}(\mu|\ |^{3r} \otimes \mu|\ |^r \otimes \mu|\ |^{-r} \otimes \mu|\ |^{-3r})$;

(4) if v is non-archimedean and π_v is special, i.e., $\pi_v = \sigma(\mu|\ |^{\frac{1}{2}}, \mu|\ |^{-\frac{1}{2}})$. Then Σ_v is a unique square integrable constituent of $\mathrm{Ind}_B^{\mathrm{GL}_4(F_v)}(\mu|\ |^{\frac{3}{2}} \otimes \mu|\ |^{\frac{1}{2}} \otimes \mu|\ |^{-\frac{1}{2}} \otimes \mu|\ |^{-\frac{3}{2}})$;

(5) if v is non-archimedean and π_v is a non-extraordinary supercuspidal representation, then $\pi_v = \pi(\chi)$, where $\pi(\chi)$ is the representation of the local Weil group induced from χ, a character of K_v, $[K_v : F_v] = 2$. Let η be the quadratic character attached to K_v/F_v by class field theory. Let χ' be the conjugate of χ by the action of the nontrivial element of the Galois group. Then Σ_v is the induced representation $\mathrm{Ind}_{\mathrm{GL}_2 \times \mathrm{GL}_2}^{\mathrm{GL}_4(F_v)} \pi(\chi^2{\chi'}^{-1}) \otimes (\pi(\chi) \otimes \eta)$;

(6) if v is non-archimedean and π_v is an extraordinary supercuspidal representation and admits a lift Σ_v, then Σ_v is either supercuspidal or is a constituent of a representation induced from a supercuspidal representation of $\mathrm{GL}_2(F_v) \times \mathrm{GL}_2(F_v)$.

Our ultimate goal is to prove that even for an extraordinary supercuspidal representation π_v, we can define the adjoint cube lift $\Sigma_v = \mathrm{Ad}^3(\pi_v)$ and $\Sigma =$

$\bigotimes_v \Sigma_v$ is an automorphic cuspidal representation of GL_4 if π is not monomial. In order to apply the converse theorem [Co-PS2], we need to consider the Rankin–Selberg L-function $L(s, \mathrm{Ad}^3(\pi) \times \pi')$ for a cuspidal representation π' of GL_2. If π_v is an extraordinary supercuspidal representation, even if we do not know whether the adjoint cube lift exists, we can still define the local L-function $L(s, \mathrm{Ad}^3(\pi_v) \times \pi_v')$ as a quotient $\frac{L(s,\sigma_v,r_1)}{L(s,\pi_v\times\pi_v')}$ in (4.1).

We have to divide into two cases: π is monomial or not.

First, we suppose π is a monomial cuspidal representation, i.e., $\pi \otimes \eta \simeq \pi$ for a nontrivial grössencharacter η. Then $\eta^2 = 1$ and η determines a quadratic extension E/F. According to [L-La], there is a grössencharacter χ of E such that $\pi = \pi(\chi)$, where $\pi(\chi)$ is the automorphic representation whose local factor at v is the one attached to the representation of the local Weil group induced from χ_v. Let χ' be the conjugate of χ by the action of the nontrivial element of the Galois group. Then the Gelbart–Jacquet lift (adjoint square) of π is given by

$$\Pi = \mathrm{Ind}_{\mathrm{GL}_2 \times \mathrm{GL}_1}^{\mathrm{GL}_3(\mathbb{A})} (\pi(\chi\chi'^{-1}) \otimes \eta).$$

There are two cases:

Case 1: $\chi\chi'^{-1}$ factors through the norm, i.e., $\chi\chi'^{-1} = \tilde{\eta} \circ N_{E/F}$ for a grössencharacter $\tilde{\eta}$ of F. Then $\pi(\chi\chi'^{-1})$ is not cuspidal. In fact, $\pi(\chi\chi'^{-1}) = \pi(\tilde{\eta}, \tilde{\eta}\eta)$. In this case,

$$L(s, \mathrm{Ad}^3(\pi) \times \pi') = L(s, (\tilde{\eta} \otimes \pi) \times \pi') L(s, (\tilde{\eta}\eta \otimes \pi) \times \pi').$$

Therefore, $L(s, \mathrm{Ad}^3(\pi) \times \pi')$ is holomorphic except possibly at $s = 0, 1$. It has a pole at $s = 1$ when $\tilde{\pi}' = \tilde{\eta} \otimes \pi$ or $\tilde{\eta}\eta \otimes \pi$.

Case 2: $\chi\chi'^{-1}$ does not factor through the norm. In this case, $\pi(\chi\chi'^{-1})$ is a cuspidal representation. Then

$$L(s, \mathrm{Ad}^3(\pi) \times \pi') = L(s, \pi(\chi\chi'^{-1}) \times \pi \times \pi').$$

By Proposition 3.10, 3.11 of [Ki-Sh2], the above triple L-function is holomorphic except possibly at $s = 0, 1$. It has a pole at $s = 1$ when $\pi' = \pi(\chi^{-2}\chi')$.

Next we assume that π is not monomial. Let Π be the adjoint square (Gelbart–Jacquet lift) of π. It is a cuspidal representation of GL_3 since π is not monomial.

Let $G = \mathrm{Spin}(2n)$ be a split spin group and

$$\theta = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-2} = e_{n-3} - e_{n-2}, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n\}.$$

Let $T \subset M_\theta = M$ be the Levi subgroup of G generated by θ and let $P = MN$ be the corresponding standard parabolic subgroup of G. Then the standard calculation (cf. [Sh4]) shows that

$$M = (\mathrm{GL}_1 \times \mathrm{SL}_{n-2} \times \mathrm{SL}_2 \times \mathrm{SL}_2)/R,$$

where

$$R = \begin{cases} \{H_{\alpha_1}(t)H_{\alpha_2}(t^2)\cdots H_{\alpha_{n-2}}(t^{n-2})H_{\alpha_{n-1}}(t^{\frac{n-2}{2}})H_{\alpha_n}(t^{\frac{n-2}{2}}) : t^{n-2} = 1\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } n \text{ even}, \\ \{H_{\alpha_1}(t^2)H_{\alpha_2}(t^4)\cdots H_{\alpha_{n-2}}(t^{2(n-2)})H_{\alpha_{n-1}}(t^{n-2})H_{\alpha_n}(t^{n-2}) : t^{2(n-2)} = 1\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } n \text{ odd}. \end{cases}$$

Let π, π' be two cuspidal representations of GL_2 and Π be a cuspidal representation of GL_{n-2}. Let π_0 (resp. π_0') denote a constituent of $\pi|_{\mathrm{SL}_2(\mathbb{A}_F)}$ (resp. $\pi|_{\mathrm{SL}_2(\mathbb{A}_F)}$). Let Π_0 be a constituent of $\Pi|_{\mathrm{SL}_n(\mathbb{A}_F)}$. If ω and ω' are central characters of π and π', then $\sigma = (\Pi_0 \otimes \omega\omega') \otimes \pi_0 \otimes \pi_0'$ can be considered as a representation of $M(\mathbb{A}_F)$. Here we need to impose a condition that if n is odd, Π has the trivial central character. This is the case $D_n - 2$ in [Sh2]. The case $n = 4$ was treated in [Ki-Sh2], yielding the holomorphy of the completed Rankin triple L-functions of $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$.

We will look at the case when $n = 5$. When we take Π to be the Gelbart–Jacquet lift of π, we obtain the L-function $L(s, \mathrm{Ad}^3(\pi) \times \pi')$. More precisely, let $G = \mathrm{Spin}(10)$. Then we have the identification $M \simeq (\mathrm{GL}_3 \times \mathrm{SL}_2 \times \mathrm{SL}_2)/\{\pm 1\}$, where $-1 = (-I_3, -I_2, -I_2)$. Then $\sigma = (\Pi \otimes \omega\omega') \otimes \pi_0 \otimes \pi_0'$ is a cuspidal representation of $M(\mathbb{A})$. Then by [Sh7, Theorem 8.2],

$$\begin{aligned} L(s, \sigma_v, r_1) &= L(s, \mathrm{Ad}^3(\pi_v) \times \pi_v')L(s, \pi_v \times \pi_v'), \\ L(s, \sigma_v, r_2) &= L(s, \Pi_v, \wedge^2 \otimes \omega_v\omega_v'). \end{aligned} \tag{4.1}$$

We can prove them by direct computation for unramified places. For ramified places, we define the L-function $L(s, \sigma_v, r_1)$ as in [Sh1, section 7]. Recall that it is defined to agree completely with Langlands' definition of L-functions whenever there is a parametrization. We define $L(s, \mathrm{Ad}^3(\pi_v) \times \pi_v')$ to be $\frac{L(s,\sigma_v,r_1)}{L(s,\pi_v\times\pi_v')}$. Therefore if $\mathrm{Ad}^3(\pi_v)$ exists (i.e., except for extraordinary supercuspidal representations), $L(s, \mathrm{Ad}^3(\pi_v) \times \pi_v')$ is a local Rankin–Selberg L-function for $\mathrm{GL}_4 \times \mathrm{GL}_2$. Using this, Shahidi proved

PROPOSITION 4.1 ([Sh7, Theorem 8.2]): *Let S be a finite set of places including all archimedean places such that π_v is spherical for $v \notin S$. Then $L_S(s, \mathrm{Ad}^3(\pi) \times \pi')$ has a meromorphic continuation to all of $\mathbb{C}$.*

Now we prove

PROPOSITION 4.2: *The normalized intertwining operator $N(s, \sigma_v, w_0)$ is holomorphic and non-zero for* $\mathrm{Re}\, s \geq \frac{1}{2}$.

Proof: If Π_v tempered, in the language of Lemma 1.4, $s\tilde{\alpha} + \Lambda_0$ is in the positive Weyl chamber for $\mathrm{Re}\, s \geq \frac{1}{2}$ since if $\pi_v = \pi(\mu| \ |^r, \mu| \ |^{-r})$, then $r < \frac{1}{4}$ by [Ge-Ja]. Hence it is enough to check conjecture 7.1 of [Sh1]: If Π_v is not supercuspidal, then in (4.1), the L-functions are L-functions of $\mathrm{GL}_k \times \mathrm{GL}_l$ and hence conjecture 7.1 of [Sh1] is satisfied. If Π_v is supercuspidal, then

$$L(s, \sigma_v, r_2) = L(s, \Pi_v, \wedge^2 \otimes \omega_v \omega_v')$$

is of the form $\prod_j (1 - \alpha_j q_v^{-s})^{-1}$ ([Sh1, Proposition 7.3]) and Proposition 1.1 applies.

Suppose Π_v is non-tempered. Then $\Pi_v = \mathrm{Ind}_B^{\mathrm{GL}_3}(\mu| \ |^a \otimes \nu \otimes \mu| \ |^{-a})$, where μ, ν are unitary characters and $0 < a < \frac{1}{2}$. Suppose $\pi_v = \pi(\mu| \ |^r, \mu| \ |^{-r})$ and $\pi_v' = \pi(\mu'| \ |^{r'}, \mu'| \ |^{-r'})$, $0 < r, r' < \frac{1}{2}$. If $\mathrm{Re}\, s \geq \frac{1}{2}$, $s - a - (r + r') > -1$, $s - a - (r - r') > -1$ and $s - x_1 > 0$. The rank-one operators are either operators for SL_2 or GL_2 and hence they are holomorphic and non-zero. Suppose π_v or π_v' is tempered. If both of them are tempered, in the language of Lemma 1.4, $s\tilde{\alpha} + \Lambda_0$ is in the positive Weyl chamber for $\mathrm{Re}\, s \geq \frac{1}{2}$ and the rank-one operators are in the situation of Proposition 1.1. Now suppose π_v is non-tempered and π_v' is tempered. The only rank-one operators left to do are for a maximal Levi which is isogenous to $\mathrm{GL}_1 \times \mathrm{SL}_2$ inside a group whose derived group is SL_3, in which case Lemma 2.10 applies, or to $\mathrm{GL}_1 \times \mathrm{SL}_2$ inside a group who se derived group is Spin(6), in which case Proposition 1.1 applies.

By Lemma 1.7, the holomorphy of $N(s, \sigma_v, w_0)$ for $\mathrm{Re}\, s \geq \frac{1}{2}$ implies the non-vanishing of $N(s, \sigma_v, w_0)$ for $\mathrm{Re}\, s \geq \frac{1}{2}$, since the standard module conjecture is valid for σ_v and we checked conjecture 7.1 of [Sh1] above in our case. ∎

LEMMA 4.3: *Let π_1, π_2 be two cuspidal representations of GL_2. Let S be the set of places where π_{1v}, π_{2v} are all tempered. Then $\underline{\delta}(S) \geq \frac{8}{10}$.*

Proof: Let S_i be the set of places where π_{iv} is tempered for $i = 1, 2, 3$. Then from [Ram], $\underline{\delta}(S_i) \geq \frac{9}{10}$ for $i = 1, 2, 3$. Let X be the set of all places. Then $\bar{\delta}(S_1 - S_2) \leq \bar{\delta}(X - S_2) \leq \frac{1}{10}$. We have $S_1 = (S_1 \cap S_2) \cup (S_1 - S_2)$. So $\underline{\delta}(S_1 \cap S_2) \geq \underline{\delta}(S_1) - \bar{\delta}(S_1 - S_2) \geq \frac{9}{10} - \frac{1}{10} = \frac{8}{10}$. ∎

PROPOSITION 4.4: *The Eisenstein series attached to* (M,σ) *is holomorphic for* $\mathrm{Re}\, s > 0$ *unless* $(\omega\omega')^2 = 1$.

Proof: Note that $w_0\sigma \cong \sigma$ if and only if $(\omega\omega')^2 = 1$. Hence our proposition follows from Corollary 1.6. ∎

PROPOSITION 4.5: $L(s,\sigma,r_2) = L(s,\Pi,\wedge^2 \otimes \omega\omega')$ *has no zeros for* $\mathrm{Re}\, s > 1$.

Proof: If $(\omega\omega')^2 \neq 1$, then by Proposition 4.4, the Eisenstein series, hence $M(s,\sigma,w_0)$, is holomorphic for $\mathrm{Re}\, s > 0$. Then by Corollary 1.6,

$$L_S(1+s,\sigma,r_1)L_S(1+2s,\sigma,r_2)$$

has no zeros for $\mathrm{Re}\, s > 0$. Consider (1.3). Since the unnormalized operators $A(s,\sigma_v,w_0)$ are non-zero operators, it follows that

$$\frac{L_S(s,\sigma,r_1)L_S(2s,\sigma,r_2)}{L_S(1+s,\sigma,r_1)L_S(1+2s,\sigma,r_2)}$$

is holomorphic for $\mathrm{Re}\, s > 0$. By [Sh3, Theorem 5.1] (the restriction has been removed in [Sh1]), $L_S(s,\sigma,r_i)$, $i = 1,2$, are holomorphic and non-zero for $\mathrm{Re}\, s > 2$. This implies that $L_S(s,\sigma,r_1)L_S(2s,\sigma,r_2)$ is holomorphic for $\mathrm{Re}\, s > 1$, hence $L_S(s,\sigma,r_1)$ is holomorphic for $\mathrm{Re}\, s > 1$. Therefore, $L_S(1+2s,\sigma,r_2)$ has no zeros for $\mathrm{Re}\, s > 0$. Our assertion follows.

Suppose $(\omega\omega')^2 = 1$. If $\omega\omega' = 1$, then our result follows from Theorem 3.1. Let $\omega\omega' \neq 1$. We give a proof, following [Sh4, Proposition 3.2]. Let E/F be the quadratic extension attached to $\omega\omega'$ and let Σ be the base change lift of Π defined by Arthur and Clozel [A-C]. Then

$$L_S(s,\Sigma,\wedge^2) = L_S(s,\Pi,\wedge^2)L_S(s,\Pi,\wedge^2 \otimes \omega\omega').$$

By [Sh4, Proposition 3.2], Σ is cuspidal. Then the left-hand side is non-vanishing for $\mathrm{Re}\, s > 1$ by Corollary 1.6 and $L_S(s,\Pi,\wedge^2)$ is holomorphic for $\mathrm{Re}\, s > 0$ by Theorem 3.1. Hence $L_S(s,\Pi,\wedge^2 \otimes \omega\omega')$ is non-vanishing for $\mathrm{Re}\, s > 1$. ∎

THEOREM 4.6: $L(s,\mathrm{Ad}^3(\pi)\times\pi')L(s,\pi\times\pi')$ *is holomorphic except possibly at* $s = 0, \frac{1}{2}, 1$.

Proof: By Proposition 4.4, if $(\omega\omega')^2 \neq 1$, then $M(s,\sigma,w_0)$ is holomorphic for $\mathrm{Re}\, s > 0$. Consider (1.2). By Proposition 4.2, the normalized operators $N(s,\sigma_v,w_0)$ have no zeros for $\mathrm{Re}\, s \geq \frac{1}{2}$. Hence $\frac{L(s,\sigma,r_1)L(2s,\sigma,r_2)}{L(1+s,\sigma,r_1)L(1+2s,\sigma,r_2)}$ is holomorphic for $s \geq \frac{1}{2}$. Pick N_0 so large that $L(1+s,\sigma,r_1)L(1+2s,\sigma,r_2)$ is holomorphic for $s \geq N_0$. Then $L(s,\sigma,r_1)L(2s,\sigma,r_2)$ is holomorphic for $s \geq N_0 - 1$. Arguing inductively, we see that $L(s,\sigma,r_1)L(2s,\sigma,r_2)$ is holomorphic for $s \geq \frac{1}{2}$. By Proposition 4.5, $L(2s,\sigma,r_2)$ has no zeros for $\operatorname{Re} s > \frac{1}{2}$. By [Sh4, Proposition 3.1], $L(2s,\sigma,r_2)$ has no zeros for $\operatorname{Re} s = \frac{1}{2}$. Hence $L(s,\sigma,r_1)$ is holomorphic for $\operatorname{Re} s \geq \frac{1}{2}$, except possibly at $s = \frac{1}{2}$. By the functional equation, we see that $L(s,\sigma,r_1)$ is holomorphic, except possibly at $s = \frac{1}{2}$.

Suppose $(\omega\omega')^2 = 1$. Then $\omega_v\omega'_v = 1$ for places of density $\frac{1}{2}$. By Lemma 4.3, there exists a place where both π_v and π'_v are unramified, tempered, and $\omega_v\omega'_v = 1$. Let $\pi_v = \pi(\mu,\nu)$ and $\pi'_v = \pi(\mu',\nu')$. Then $\omega_v = \mu\nu$ and $\omega'_v = \mu'\nu'$ and $\Pi_v = \pi(\mu\nu^{-1}, 1, \mu^{-1}\nu)$. Then under the isogeny $\mathrm{Spin}(10) \longmapsto \mathrm{SO}_{10}$, $I(s,\sigma_v)$ corresponds to an induced representation

$$\mathrm{Ind}_{\mathrm{GL}_3\times \mathrm{SO}_4}^{\mathrm{SO}_{10}}(|\det|^s\pi_1\otimes\pi_2),$$

where $\pi_1 = \mathrm{Ind}_B^{\mathrm{GL}_3}(\eta_1\otimes\eta_2\otimes\eta_3)$, $\eta_1^2 = \omega_v\omega'_v\mu^2\nu^{-2}$, $\eta_2^2 = \omega_v\omega'_v$, $\eta_3^2 = \omega_v\omega'_v\mu^{-2}\nu^2$, and π_2 is the unique generic component of $\mathrm{Ind}_B^{\mathrm{SO}_4}(\eta_4\otimes\eta_5)$, $\eta_4^2 = \mu\nu^{-1}\mu'\nu'^{-1}$, $\eta_5^2 = \mu^{-1}\nu\mu'\nu'^{-1}$. Note that π_1 has the trivial central character. By Lemma 3.6, $J(s,\sigma_v)$ is not unitary if $s > \frac{1}{2}$, $s \neq 1$. Therefore $L(s,\mathrm{Ad}^3(\pi)\times\pi')L(s,\pi\times\pi')$ cannot have a pole for $s > \frac{1}{2}$, $s \neq 1$. We now apply the functional equation. ∎

CONJECTURE 4.7: *$L(s,\mathrm{Ad}^3(\pi)\times\pi')$ is holomorphic at $s = \frac{1}{2}$.*

CONJECTURE 4.8: *$L(s,\mathrm{Ad}^3(\pi)\times\pi')$ is entire if π is not monomial.*

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